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TENSOR RAYLEIGH QUOTIENT ITERATIVE METHOD FOR THREE-PARAMETER EIGENVALUE PROBLEMS

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ABSTRACT

Three-parameter eigenvalue problems are discussed in this paper. Tensor Rayleigh Quotient Iterative Method for Three-Parameter eigenvalue problems are discussed in this paper. Finally some numerical results are presented to illustrate the performance and application of this method.

KEYWORDS: Multiparameter, Eigenvalue, Eigenvector, Tensor Rayleigh Quotient

1.1 INTRODUCTION

Multiparameter eigenvalue problems are generalization of one-parameter eigenvalue problems and can be found when the method of separation of variables is applied to certain boundary value problems associated with partial differential equations. Much more works have been done in the field of one-parameter eigenvalue problems, both theoretically and numerically compared to two-parameter or more than two-parameter eigenvalue problems. Some works have been done theoretically in the field of multiparameter eigenvalue problems. Few authors have dealt with the multiparameter eigenvalue problems numerically mainly in two-parametic cases. Numerical methods applied to a three-parameter problems are very limited and hence some contribution in this area are always in needed.

1.2 THREE-PARAMETER EIGENVALUE PROBLEM AND ITS REDUCTION TO A SYSTEM OF ONE-PARAMETER PROBLEMS

Consider the three-parameter eigenvalue problems

$$A_{10}x = \lambda_1 A_{11}x + \lambda_2 A_{12}x + \lambda_3 A_{13}x$$

$$A_{20}y = \lambda_1 A_{21}y + \lambda_2 A_{22}y + \lambda_3 A_{23}y$$

$$A_{30}z = \lambda_1 A_{31}z + \lambda_2 A_{32}z + \lambda_3 A_{33}z \tag{1.2.1}$$

Where $\lambda_i \in \Box$, i=1,2,3 and

$$x \in \mathbb{C}^{n} \setminus \{0\}, A_{10}, A_{11}, A_{12}, A_{13} \in \mathbb{C}^{n\times n}$$
$$y \in \mathbb{C}^{m} \setminus \{0\}, A_{20}, A_{21}, A_{22}, A_{23} \in \mathbb{C}^{m\times m}$$
$$z \in \mathbb{C}^{p} \setminus \{0\}, A_{30}, A_{31}, A_{32}, A_{33} \in \mathbb{C}^{p\times p}$$

Where $\lambda_i \in \mathbb{C}$, i=1,2,3 are called the eigenvalues and x, y, z are called eigenvectors of the problem.

Problem (1.2.1) can be reduced to a system of three one-parameter problems:

$$\begin{aligned} \Delta_1 u &= \lambda_1 \Delta_0 u \\ \Delta_2 u &= \lambda_2 \Delta_0 u \\ \Delta_3 u &= \lambda_3 \Delta_0 u \end{aligned} \tag{1.2.2}$$

where $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ are $(mnp)\times(mnp)$ dimensional matrices defined as

$$\Delta_0 = A_{11}\otimes A_{22}\otimes A_{33} - A_{11}\otimes A_{23}\otimes A_{32} + A_{12}\otimes A_{23}\otimes A_{31} - A_{12}\otimes A_{21}\otimes A_{33} \tag{1.2.3}$$

$$+A_{13}\otimes A_{21}\otimes A_{32} - A_{13}\otimes A_{22}\otimes A_{31}$$

$$\Delta_1 = A_{10}\otimes A_{22}\otimes A_{33} - A_{10}\otimes A_{23}\otimes A_{32} + A_{12}\otimes A_{23}\otimes A_{30} - A_{12}\otimes A_{20}\otimes A_{33} \tag{1.2.4}$$

$$+A_{13}\otimes A_{20}\otimes A_{32} - A_{13}\otimes A_{22}\otimes A_{30}$$

$$\Delta_2 = A_{11}\otimes A_{20}\otimes A_{33} - A_{11}\otimes A_{23}\otimes A_{30} + A_{10}\otimes A_{23}\otimes A_{31} - A_{10}\otimes A_{21}\otimes A_{33}$$

$$+A_{13}\otimes A_{21}\otimes A_{30} - A_{13}\otimes A_{20}\otimes A_{31} \tag{1.2.5}$$

$$\Delta_3 = A_{11}\otimes A_{22}\otimes A_{30} - A_{11}\otimes A_{20}\otimes A_{32} + A_{12}\otimes A_{20}\otimes A_{31} - A_{12}\otimes A_{21}\otimes A_{30}$$

$$+A_{10}\otimes A_{21}\otimes A_{32} - A_{10}\otimes A_{22}\otimes A_{31} \tag{1.2.6}$$

And

$$u = x\otimes y\otimes z$$

Theorem : Let $(\lambda_1, \lambda_2, \lambda_3)$ be an eigenvalue and (x, y, z) a corresponding eigenvector of the system (1.2.1) then $(\lambda_1, \lambda_2, \lambda_3)$ is an eigenvalue of the system (1.2.2) and $u = x\otimes y\otimes z$ is the corresponding eigenvector.

Definition 1.3.1. The Kronecker product $(.\otimes.) : \mathbb{C}^{m\times n}\times\mathbb{C}^{p\times q} \to \mathbb{C}^{mp\times nq}$ is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$$

Where we use the standard notation $(A)_{ij} = a_{ij}$

The Kronecker product is a special case of the tensor product, and as such it inherits the properties of bilinearity and associativity, i.e.

$$(\mathrm{kA}) \otimes \mathrm{B} = \mathrm{A} \otimes (\mathrm{kB}) = \mathrm{k}(\mathrm{A} \otimes \mathrm{B})$$

$$\mathrm{A} \otimes (\mathrm{B} + \mathrm{C}) = \mathrm{A} \otimes \mathrm{B} + \mathrm{A} \otimes C$$

$$(\mathrm{A} + \mathrm{B}) \otimes \mathrm{C} = \mathrm{A} \otimes C + B \otimes C$$

We now establish a famous property of the Kronecker product, from [9].

Lemma (Mixed product property). Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}, C \in \mathbb{R}^{n \times k}, D \in \mathbb{R}^{q \times r}$. Then

$$(A \otimes B)(\mathrm{C} \otimes \mathrm{D}) = (\mathrm{AC} \otimes \mathrm{BD}).$$

In particular, if $A, B \in \mathbb{R}^{m \times m}$ and $\mathrm{x}, \mathrm{y} \in \mathbb{R}^{m}$ then

$$(A \otimes B)(\mathrm{x} \otimes \mathrm{y}) = \mathrm{Ax} \otimes \mathrm{By}.$$

2.1 TENSOR RAYLEIGH QUOTIENT

The Tensor Rayleigh Quotient $\rho(x, y, z, A_{10}, A_{11}, A_{12}, A_{13}, A_{20}, A_{21}, A_{22}, A_{23}, A_{30}, A_{31}, A_{32}, A_{33})$ is an triplet (ρ_1, ρ_2, ρ_3) such that

$$\rho_1 = \frac{u^T \Delta_1 u}{u^T \Delta_0 u}$$

$$\rho_2 = \frac{u^T \Delta_2 u}{u^T \Delta_0 u}$$

$$\rho_3 = \frac{u^T \Delta_3 u}{u^T \Delta_0 u}$$

Where $u = x \otimes y \otimes z$

Hence the Tensor Rayleigh Quotient at an exact eigenvector is

$$\rho_1 = \frac{u^T \Delta_1 u}{u^T \Delta_0 u} = \frac{\lambda_1 u^T \Delta_0 u}{u^T \Delta_0 u} = \lambda_1$$

$$\rho_2 = \frac{u^T \Delta_2 u}{u^T \Delta_0 u} = \frac{\lambda_2 u^T \Delta_0 u}{u^T \Delta_0 u} = \lambda_2$$

$$\rho_3 = \frac{u^T \Delta_3 u}{u^T \Delta_0 u} = \frac{\lambda_3 u^T \Delta_0 u}{u^T \Delta_0 u} = \lambda_3$$

Now

$$\begin{aligned} \mathrm{u}^T \Delta_0 u = u^T (&A_{11} \otimes A_{22} \otimes A_{33} - A_{11} \otimes A_{23} \otimes A_{32} + A_{12} \otimes A_{23} \otimes A_{31} - A_{12} \otimes A_{21} \otimes A_{33} \\ &+ A_{13} \otimes A_{21} \otimes A_{32} - A_{13} \otimes A_{22} \otimes A_{31})\, \mathrm{u} \end{aligned}$$

$$\begin{aligned} \mathrm{u}^T \Delta_0 u = (x^T \otimes y^T \otimes z^T)(&A_{11} \otimes A_{22} \otimes A_{33} - A_{11} \otimes A_{23} \otimes A_{32} + A_{12} \otimes A_{23} \otimes A_{31} - A_{12} \otimes A_{21} \otimes A_{33} \\ &+ A_{13} \otimes A_{21} \otimes A_{32} - A_{13} \otimes A_{22} \otimes A_{31})(\mathrm{x} \otimes \mathrm{y} \otimes \mathrm{z}) \end{aligned}$$

$$\begin{aligned} &= x^T A_{11} x \otimes y^T A_{22} y \otimes z^T A_{33} z - x^T A_{11} x \otimes y^T A_{23} y \otimes z^T A_{32} z + x^T A_{12} x \otimes y^T A_{23} y \otimes z^T A_{31} z - x^T A_{12} x \otimes y^T A_{21} y \otimes z^T A_{33} z \\ &+ x^T A_{13} x \otimes y^T A_{21} y \otimes z^T A_{32} z - x^T A_{13} x \otimes y^T A_{22} y \otimes z^T A_{31}\, \mathrm{z}) \end{aligned}$$

But $\mathrm{x}^T A_{11} x$, $y^T A_{11} y$ etc are scalars. Hence $\otimes$ becomes normal multiplication. So

$$\begin{aligned} &\mathrm{u}^T \Delta_0 u \\ &= (x^T A_{11} x)(y^T A_{22} y)(z^T A_{33} z) - (x^T A_{11} x)(y^T A_{23} y)(z^T A_{32} z) + (x^T A_{12} x)(y^T A_{23} y)(z^T A_{31} z) - (x^T A_{12} x)(y^T A_{21} y)(z^T A_{33} z) \\ &+ (x^T A_{13} x)(y^T A_{21} y)(z^T A_{32} z) - (x^T A_{13} x)(y^T A_{22} y)(z^T A_{31}\, \mathrm{z}) \end{aligned}$$

Similarly

$$\begin{aligned} u^T \Delta_1 u &= (x^T A_{10} x)(y^T A_{22} y)(z^T A_{33} z) - (x^T A_{10} x)(y^T A_{23} y)(z^T A_{32} z) + (x^T A_{12} x)(y^T A_{23} y)(z^T A_{30} z) - (x^T A_{12} x)(y^T A_{20} y)(z^T A_{33} z) \\ &+ (x^T A_{13} x)(y^T A_{20} y)(z^T A_{32} z) - (x^T A_{13} x)(y^T A_{22} y)(z^T A_{30}\, \mathrm{z}) \end{aligned}$$

$$\begin{aligned} u^T \Delta_2 u &= (x^T A_{11} x)(y^T A_{20} y)(z^T A_{33} z) - (x^T A_{11} x)(y^T A_{23} y)(z^T A_{30} z) + (x^T A_{10} x)(y^T A_{23} y)(z^T A_{31} z) - (x^T A_{10} x)(y^T A_{21} y)(z^T A_{33} z) \\ &+ (x^T A_{13} x)(y^T A_{21} y)(z^T A_{30} z) - (x^T A_{13} x)(y^T A_{20} y)(z^T A_{31}\, \mathrm{z}) \end{aligned}$$

$$\begin{aligned} u^T \Delta_3 u &= (x^T A_{11} x)(y^T A_{22} y)(z^T A_{30} z) - (x^T A_{11} x)(y^T A_{20} y)(z^T A_{32} z) + (x^T A_{12} x)(y^T A_{20} y)(z^T A_{31} z) - (x^T A_{12} x)(y^T A_{21} y)(z^T A_{30} z) \\ &+ (x^T A_{10} x)(y^T A_{21} y)(z^T A_{32} z) - (x^T A_{10} x)(y^T A_{22} y)(z^T A_{31}\, \mathrm{z}) \end{aligned}$$

2.1 TENSOR RAYLEIGH QUOTIENT ITERATIVE METHOD

Consider the three parameter eigenvalue problem (1.2.1). In mtrix form it can be written as

$$F(u)=\begin{bmatrix} A_{10}x - A_{11}\lambda_1 x - A_{12}\lambda_2 x - A_{13}\lambda_3 x \\ A_{20}y - A_{21}\lambda_1 y - A_{22}\lambda_2 y - A_{23}\lambda_3 y \\ A_{30}z - A_{31}\lambda_1 z - A_{32}\lambda_2 z - A_{33}\lambda_3 z \\ \frac{1}{2}x^T x - 1 \\ \frac{1}{2}y^T y - 1 \\ \frac{1}{2}z_k^T z_k - 1 \end{bmatrix}$$

Newton's method applies to the equation F(u)=0 gives

$$F^{/}(u_k)(u_{k+1}-u_k) = -F(u_k)$$

Where

$$F^{/}(u)=\begin{bmatrix} A_{10}-\lambda_1 A_{11}-\lambda_2 A_{12}-\lambda_3 A_{13} & 0 & 0 & -A_{11}x & -A_{12}x & -A_{13}x \\ 0 & A_{20}-\lambda_1 A_{21}-\lambda_2 A_{22}-\lambda_3 A_{23} & 0 & -A_{21}y & -A_{22}y & -A_{23}y \\ 0 & 0 & A_{30}-\lambda_1 A_{31}-\lambda_2 A_{32}-\lambda_3 A_{33} & -A_{31}z & -A_{32}z & -A_{33}z \\ \frac{1}{2}x^T & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}y^T & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}z^T & 0 & 0 & 0 \end{bmatrix}$$

Now to apply Tensor Rayleigh Quotient Iterative method for three-parameter eigenvalue problem we replace respectively $\lambda_1, \lambda_2, \lambda_3$ with ρ_1, ρ_2, ρ_3 in equation (1.2.1). Let (x_0, y_0, z_0) be the eigenvectors where $\|x_0\| = 1, \|y_0\| = 1, \|z_0\| = 1$ for the problem (1.2.1). An initial approximation for the eigenvalue is not needed. Since we can calculate one using the Tensor Rayleigh Quotient.

Now we have to solve

F(u)=0 where

$$F(u)=\begin{bmatrix} A_{10}x - A_{11}\rho_1 x - A_{12}\rho_2 x - A_{13}\rho_3 x \\ A_{20}y - A_{21}\rho_1 y - A_{22}\rho_2 y - A_{23}\rho_3 y \\ A_{30}z - A_{31}\rho_1 z - A_{32}\rho_2 z - A_{33}\rho_3 z \\ \frac{1}{2}x^T x - 1 \\ \frac{1}{2}y^T y - 1 \\ \frac{1}{2}z_k^T z_k - 1 \end{bmatrix}$$

Using Newton's method we have

$$F^{/}(u_k)(u_{k+1}-u_k)=-F(u_k) \tag{2.1.1}$$

Where

$$F^{/}(u)=\begin{bmatrix} A_{10}-\rho_1A_{11}-\rho_2A_{12}-\rho_3A_{13} & 0 & 0 & -A_{11}x & -A_{12}x & -A_{13}x \\ 0 & A_{20}-\rho_1A_{21}-\rho_2A_{22}-\rho_3A_{23} & 0 & -A_{21}y & -A_{22}y & -A_{23}y \\ 0 & 0 & A_{30}-\rho_1A_{31}-\rho_2A_{32}-\rho_3A_{33} & -A_{31}z & -A_{32}z & -A_{33}z \\ \frac{1}{2}x^T & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}y^T & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}z^T & 0 & 0 & 0 \end{bmatrix}$$

Thus (2.1.1) $\Rightarrow$

$$\begin{bmatrix} A_{10}-\rho_1^{(k)}A_{11}-\rho_2^{(k)}A_{12}-\rho_3^{(k)}A_{13} & 0 & 0 & -A_{11}x_k & -A_{12}x_k & -A_{13}x_k \\ 0 & A_{20}-\rho_1^{(k)}A_{21}-\rho_2^{(k)}A_{22}-\rho_3^{(k)}A_{23} & 0 & -A_{21}y_k & -A_{22}y_k & -A_{23}y_k \\ 0 & 0 & A_{30}-\rho_1^{k}A_{31}-\rho_2^{k}A_{32}-\rho_3^{k}A_{33} & -A_{31}z_k & -A_{32}z_k & -A_{33}z_k \\ \frac{1}{2}x_k^{T} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}y_k^{T} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}z_k^{T} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \Delta x_k \\ \Delta y_k \\ \Delta z_k \\ \Delta\lambda_1^k \\ \Delta\lambda_2^k \\ \Delta\lambda_3^k \end{bmatrix}=-$$

$$\begin{bmatrix} A_{10}x_k-A_{11}\rho_1^{(k)}x_k-A_{12}\rho_2^{(k)}x_k-A_{13}\rho_3^{(k)}x_k \\ A_{20}y_k-A_{21}\rho_1^{(k)}y_k-A_{22}\rho_2^{(k)}y_k-A_{23}\rho_3^{(k)}y_k \\ A_{30}z_k-A_{31}\rho_1^{(k)}z_k-A_{32}\rho_2^{(k)}z_k-A_{33}\rho_3^{(k)}z_k \\ \frac{1}{2}x_k^{T}x_k-1 \\ \frac{1}{2}y_k^{T}y_k-1 \\ \frac{1}{2}z_k^{T}z_k-1 \end{bmatrix} \tag{2.1.2}$$

Consider the 1st component of (2.1.2)

$$(A_{10}-\rho_1^{(K)}A_{11}-\rho_2^{(K)}A_{12}-\rho_3^{(K)}A_{13})(\mathrm{x}_k+\Delta x_k)=A_{11}x_k\Delta\rho_1^{(k)}+A_{12}x_k\Delta\rho_2^{(k)}+A_{13}x_k\Delta\rho_3^{(k)} \tag{2.1.3}$$

Consider the 2nd component of (2.1.2)

$$(A_{20}-\rho_1^{(K)}A_{21}-\rho_2^{(K)}A_{22}-\rho_3^{(K)}A_{23})(y_k+\Delta y_k)=A_{21}y_k\Delta\rho_1^{(k)}+A_{22}y_k\Delta\rho_2^{(k)}+A_{23}y_k\Delta\rho_3^{(k)} \quad (2.1.4)$$

Consider the 3rd component of (2.1.2)

$$(A_{30}-\rho_1^{(K)}A_{31}-\rho_2^{(K)}A_{32}-\rho_3^{(K)}A_{33})(z_k+\Delta z_k)=A_{31}z_k\Delta\rho_1^{(k)}+A_{32}z_k\Delta\rho_2^{(k)}+A_{33}z_k\Delta\rho_3^{(k)} \quad (2.1.5)$$

Consider the 4th component of (2.1.2)

$$x_k^T x_{k+1}=2x_k^{\ T}x_k-1 \quad (2.1.6)$$

Similarly from the 5th and 6th component

$$y_k^T y_{k+1}=2y_k^{\ T}y_k-1 \quad (2.1.7)$$

$$z_k^T z_{k+1}=2z_k^{\ T}z_k-1 \quad (2.1.8)$$

From (2.1.3)

$$x_{k+1}=(A_{10}-\rho_1^{(K)}A_{11}-\rho_2^{(K)}A_{12}-\rho_3^{(K)}A_{13})^{-1}(A_{11}x_k\Delta\rho_1^{(k)}+A_{12}x_k\Delta\rho_2^{(k)}+A_{13}x_k\Delta\rho_3^{(k)}) \quad (2.1.9)$$

Similarly

$$y_{k+1}=(A_{20}-\rho_1^{(K)}A_{21}-\rho_2^{(K)}A_{22}-\rho_3^{(K)}A_{23})^{-1}(A_{21}y_k\Delta\rho_1^{(k)}+A_{22}y_k\Delta\rho_2^{(k)}+A_{23}y_k\Delta\rho_3^{(k)}) \quad (2.1.10)$$

$$z_{k+1}=(A_{30}-\rho_1^{(K)}A_{31}-\rho_2^{(K)}A_{32}-\rho_3^{(K)}A_{33})^{-1}(A_{31}z_k\Delta\rho_1^{(k)}+A_{32}z_k\Delta\rho_2^{(k)}+A_{33}z_k\Delta\rho_3^{(k)}) \quad (2.1.11)$$

Multiplying (2.1.9), (2.1.10), (2.1.11) on the left by x_k^T, y_k^T, z_k^T and using (2.1.6), (2.1.7), (2.1.8)

$$\begin{bmatrix} x_k^{\ T}u_k & x_k^{\ T}v_k & x^{\ T}w_k \\ y_k^{\ T}p_k & y_k^{\ T}q_k & y_k^{\ T}r_k \\ z_k^{\ T}a_k & z_k^{\ T}b_k & z_k^{\ T}c_k \end{bmatrix}\begin{bmatrix}\Delta\rho_1^{(k)}\\ \Delta\rho_2^{(k)}\\ \Delta\rho_3^{(k)}\end{bmatrix}=\begin{bmatrix}2x_k^{\ T}x_k-1\\ 2y_k^{\ T}y_k-1\\ 2z_k^{\ T}z_k-1\end{bmatrix}$$

Where

$$u_k=(A_{10}-\rho_1^{(K)}A_{11}-\rho_2^{(K)}A_{12}-\rho_3^{(K)}A_{13})^{-1}A_{11}x_k$$

$$v_k=(A_{10}-\rho_1^{(K)}A_{11}-\rho_2^{(K)}A_{12}-\rho_3^{(K)}A_{13})^{-1}A_{12}x_k$$

$$w_k=(A_{10}-\rho_1^{(K)}A_{11}-\rho_2^{(K)}A_{12}-\rho_3^{(K)}A_{13})^{-1}A_{13}x_k$$

$$p_k=(A_{20}-\rho_1^{(K)}A_{21}-\rho_2^{(K)}A_{22}-\rho_3^{(K)}A_{23})^{-1}A_{21}y_k$$

$$q_k = (A_{20} - \rho_1^{(K)} A_{21} - \rho_2^{(\mathrm{K})} A_{22} - \rho_3^{(K)} A_{23})^{-1} A_{22} y_k$$

$$r_k = (A_{20} - \rho_1^{(K)} A_{21} - \rho_2^{(\mathrm{K})} A_{22} - \rho_3^{(K)} A_{23})^{-1} A_{23} y_k$$

$$a_k = (A_{30} - \rho_1^{(K)} A_{31} - \rho_2^{(\mathrm{K})} A_{32} - \rho_3^{(K)} A_{33})^{-1} A_{31} z_k$$

$$b_k = (A_{30} - \rho_1^{(K)} A_{31} - \rho_2^{(\mathrm{K})} A_{32} - \rho_3^{(K)} A_{33})^{-1} A_{32} z_k$$

$$c_k = (A_{30} - \rho_1^{(K)} A_{31} - \rho_2^{(\mathrm{K})} A_{32} - \rho_3^{(K)} A_{33})^{-1} A_{33} z_k$$

The following algorithm can be used for Tensor Rayleigh Quotient Iterative Method

- Start with an initial approximations $(\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0)$ to the eigenvectors where $\|x_0\| = 1, \|y_0\| = 1, \|z_0\| = 1$ Then Calculate the Tensor Rayleigh Quotient $(\rho_1{}^k, \rho_2{}^k, \rho_3{}^k)$
- Check determinant of matrices $[A_{10} - \rho_1^k A_{11} - \rho_2^k A_{12} - \rho_3^k A_{13}], [A_{20} - \rho_1^k A_{21} - \rho_2^k A_{22} - \rho_3^k A_{23}], [A_{30} - \rho_1^k A_{31} - \rho_2^k A_{32} - \rho_3^k A_{33}]$

and if either are equal to 0, perturb $(\rho_1{}^k, \rho_2{}^k, \rho_3{}^k)$ slightly.

- Solve the following equations

$$[A_{10} - \rho_1^{(k)} A_{11} - \rho_2^{(k)} A_{12} - \rho_3^{(k)} A_{13}]\mathrm{u}_k = \mathrm{A}_{11}\mathrm{x}_k$$
$$[A_{10} - \rho_1^{(k)} A_{11} - \rho_2^{(k)} A_{12} - \rho_3^{(k)} A_{13}]\mathrm{v}_k = \mathrm{A}_{12}\mathrm{x}_k$$
$$[A_{10} - \rho_1^{(k)} A_{11} - \rho_2^{(k)} A_{12} - \rho_3^{(k)} A_{13}]\mathrm{w}_k = \mathrm{A}_{13}\mathrm{x}_k$$

$$[A_{20} - \rho_1^{(k)} A_{21} - \rho_2^{(k)} A_{22} - \rho_3^{(k)} A_{23}]\mathrm{p}_k = A_{21} y_k$$
$$[A_{20} - \rho_1^{(k)} A_{21} - \rho_2^{(k)} A_{22} - \rho_3^{(k)} A_{23}]\mathrm{q}_k = A_{22} y_k$$
$$[A_{20} - \rho_1^{(k)} A_{21} - \rho_2^{(k)} A_{22} - \rho_3^{(k)} A_{23}]\mathrm{r}_k = A_{23} y_k$$

$$[A_{30} - \rho_1^{(k)} A_{31} - \rho_2^{(k)} A_{32} - \rho_3^{(k)} A_{33}]\mathrm{a}_k = A_{31} z_k$$
$$[A_{30} - \rho_1^{(k)} A_{31} - \rho_2^{(k)} A_{32} - \rho_3^{(k)} A_{33}]\mathrm{b}_k = A_{32} z_k$$
$$[A_{30} - \rho_1^{(k)} A_{31} - \rho_2^{(k)} A_{32} - \rho_3^{(k)} A_{33}]\mathrm{c}_k = A_{33} z_k$$

- Set up and solve the following system

$$\begin{bmatrix} x_k{}^T u_k & x_k{}^T v_k & x_k{}^T w_k \\ y_k{}^T p_k & y_k{}^T q_k & y_k{}^T r_k \\ z_k{}^T a_k & z_k{}^T b_k & z_k{}^T c_k \end{bmatrix} \begin{bmatrix} \Delta\rho_1^{(k)} \\ \Delta\rho_2^{(k)} \\ \Delta\rho_3^{(k)} \end{bmatrix} = \begin{bmatrix} 2x_k{}^T x_k - 1 \\ 2y_k{}^T y_k - 1 \\ 2z_k{}^T z_k - 1 \end{bmatrix}$$

- Update the approximate eigenvectors

$$x_{k+1} = \Delta\rho_1^{(k)} u_k + \Delta\rho_2^{(K)} v_k + \Delta\rho_3^{(k)} w_k$$
$$y_{k+1} = \Delta\rho_1^{(k)} p_k + \Delta\rho_2^{(K)} q_k + \Delta\rho_3^{(k)} r_k$$
$$z_{k+1} = \Delta\rho_1^{(k)} a_k + \Delta\rho_2^{(K)} b_k + \Delta\rho_3^{(k)} c_k$$

- Normalise the approximated eigenvectors

$$x_{k+1} = \frac{x_{k+1}}{\|x_{k+1}\|}$$

$$y_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|}$$

$$z_{k+1} = \frac{z_{k+1}}{\|z_{k+1}\|}$$

3.1 NUMERICAK EXAMPLE

We now present a numerical example to show the behaviour and application of our method

Consider the three parameter eigenvalue problem

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 7 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 8 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 14 & 0 \\ 0 & 13 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 30 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 75 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 57 & 0 \\ 0 & 14 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Table 1

Starting Eigenvector	Iteration	$\rho^{(K)} = (\rho_1^{(K)}, \rho_2^{(K)}, \rho_3^{(K)})^T$	$\left\|\rho^{(K+1)} - \rho^{(K)}\right\|$
$x_0 = \begin{pmatrix} 1 \\ -.05 \end{pmatrix}$ $y_0 = \begin{pmatrix} 1 \\ -.05 \end{pmatrix}$ $z_0 = \begin{pmatrix} 1 \\ -.05 \end{pmatrix}$	0	$\begin{pmatrix} -.0022 \\ -.1501 \\ .2514 \end{pmatrix}$	
	1	$\begin{pmatrix} -8.9850e-009 \\ -.1500 \\ .2500 \end{pmatrix}$	.0711

	2	$\begin{pmatrix} -1.1924e-016 \\ -.1500 \\ .2500 \end{pmatrix}$	.0025
$x_0 = \begin{pmatrix} 1.5 \\ .5 \end{pmatrix}$ $y_0 = \begin{pmatrix} -.05 \\ 1 \end{pmatrix}$ $z_0 = \begin{pmatrix} .05 \\ 1.25 \end{pmatrix}$	0	$\begin{pmatrix} -.2469 \\ -.0340 \\ .2872 \end{pmatrix}$	
	1	$\begin{pmatrix} -.2494 \\ -5.3092e-004 \\ .2503 \end{pmatrix}$	.0506
	2	$\begin{pmatrix} -.25 \\ -3.0522e-016 \\ .25 \end{pmatrix}$	8.5550e-004
	3	$\begin{pmatrix} -.25 \\ -1.8499e-016 \\ .25 \end{pmatrix}$	1.2023e-016
	4	$\begin{pmatrix} -.25 \\ -1.8499e-016 \\ .25 \end{pmatrix}$	7.4470e-017
$x_0 = \begin{pmatrix} 2 \\ -.5 \end{pmatrix}$ $y_0 = \begin{pmatrix} 1.5 \\ .05 \end{pmatrix}$ $z_0 = \begin{pmatrix} -.05 \\ 1.5 \end{pmatrix}$	0	$\begin{pmatrix} -.0524 \\ -.0182 \\ .1858 \end{pmatrix}$	
	1	$\begin{pmatrix} -9.8358e-005 \\ -3.0194e-005 \\ .1429 \end{pmatrix}$	.0700
	2	$\begin{pmatrix} -6.0964e-016 \\ 1.5241e-016 \\ .1429 \end{pmatrix}$	1.0289e-004
	3	$\begin{pmatrix} 3.5141e-016 \\ 0 \\ .1429 \end{pmatrix}$	9.7306e-016

	4	$\begin{pmatrix} 2.07970e-016 \\ 0 \\ .1429 \end{pmatrix}$	1.4344e-016
$x_0 = \begin{pmatrix} .02 \\ 1.7 \end{pmatrix}$ $y_0 = \begin{pmatrix} .05 \\ 2 \end{pmatrix}$ $z_0 = \begin{pmatrix} 1.2 \\ -.08 \end{pmatrix}$	0	$\begin{pmatrix} -.1860 \\ -.3401 \\ .5981 \end{pmatrix}$	
	1	$\begin{pmatrix} -.1856 \\ -.3402 \\ .5979 \end{pmatrix}$	4.5826e-004
	2	$\begin{pmatrix} -.1856 \\ -.3402 \\ .5979 \end{pmatrix}$	0
$x_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $y_0 = \begin{pmatrix} 1.7 \\ -.05 \end{pmatrix}$ $z_0 = \begin{pmatrix} .02 \\ 1.5 \end{pmatrix}$	0	$\begin{pmatrix} -1.1778 \\ -.3929 \\ 1.0967 \end{pmatrix}$	
	1	$\begin{pmatrix} -1.5669 \\ -.5222 \\ 1.4115 \end{pmatrix}$	.5169
	2	$\begin{pmatrix} -1.7946 \\ -.5982 \\ 1.5956 \end{pmatrix}$	.3025
	3	$\begin{pmatrix} -1.8 \\ -.6 \\ 1.6 \end{pmatrix}$	.0072
	4	$\begin{pmatrix} -1.8 \\ -.6 \\ 1.6 \end{pmatrix}$	0

$x_0 = \begin{pmatrix} .01 \\ 1.2 \end{pmatrix}$ $y_0 = \begin{pmatrix} .03 \\ 1.5 \end{pmatrix}$ $z_0 = \begin{pmatrix} .2 \\ .8 \end{pmatrix}$	0	$\begin{pmatrix} -.2271 \\ -.3205 \\ .6039 \end{pmatrix}$	
	1	$\begin{pmatrix} -.2361 \\ -.3161 \\ .6052 \end{pmatrix}$	.0101
	2	$\begin{pmatrix} -.2361 \\ -.3161 \\ .6052 \end{pmatrix}$	0
$x_0 = \begin{pmatrix} -.05 \\ 1.5 \end{pmatrix}$ $y_0 = \begin{pmatrix} 1.6 \\ -.01 \end{pmatrix}$ $z_0 = \begin{pmatrix} 1.5 \\ -.01 \end{pmatrix}$	0	$\begin{pmatrix} -1.6329 \\ -.2725 \\ 1.2706 \end{pmatrix}$	
	1	$\begin{pmatrix} -1.6364 \\ -.2727 \\ 1.2727 \end{pmatrix}$	.0041
	2	$\begin{pmatrix} -1.6364 \\ -.2727 \\ 1.2727 \end{pmatrix}$	0

Conclusion: Table 1.3 shows that the successive difference between the eigenvalues are gradually decreases. So the method converges to exact solution rapidly. In this method the approximate eigenvalues are obtained easily. So one can use this method easily to solve three-parameter eigenvalue problems. Here the approximated eigenvalues are $(-1.1924e-016,-.1500,.2500)^T,(-.25,-1.8499e-016,.25)^T,(2.07970e-016,0,.1429)^T,(-.1856,-.3402,.5979)^T,(-1.8,-.6,1.6)^T,(-1.6364,-.2727,1.2727)^T,(-.2361,-.3161,.6052)^T$.

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